

Module 11

Section 5.3

Trigonometric Functions of Any Angle

Math 130: Advanced Technical Mathematics | Unit 3

Section 5.3

Trigonometric Functions of Any Angle

8 ALEKS Skills

Section 5.3 — Skills Overview

1

Sketching angle $|\theta| < 360^\circ$ and its reference angle

2

Reference angles in degrees: Problem type 1

3

Sketching angle $|\theta| > 360^\circ$ and its reference angle

4

Reference angles in degrees: Problem type 2

5

Sketching angle $|\theta| < 2\pi$ and its reference angle

6

Reference angles in radians: Problem type 1

7

Determining terminal point from trig sign info

8

Finding trig values given angle information

Trig Functions of Any Angle

General Definition: For any angle θ with point $P(x, y)$ on its terminal side at distance r from the origin:

$$\sin \theta = y / r$$

$$\csc \theta = r / y$$

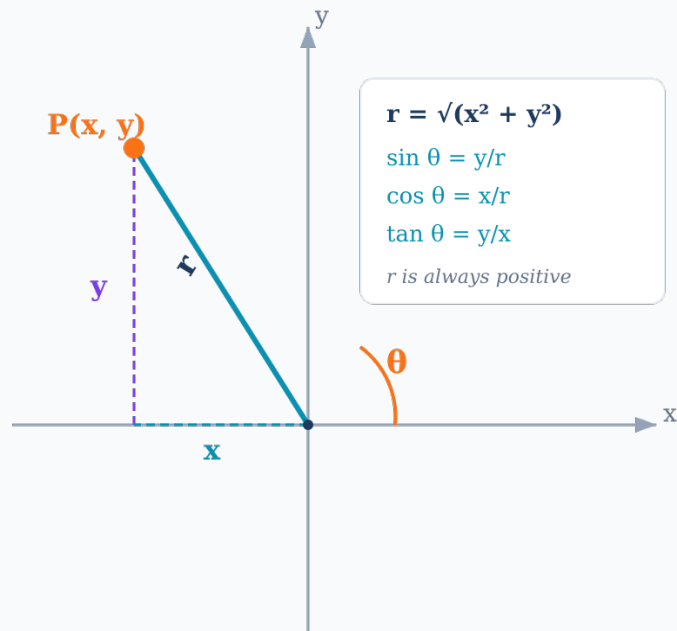
$$\cos \theta = x / r$$

$$\sec \theta = r / x$$

$$\tan \theta = y / x$$

$$\cot \theta = x / y$$

$r = \sqrt{x^2 + y^2}$ is always positive — signs depend on x and y



Reference Angles

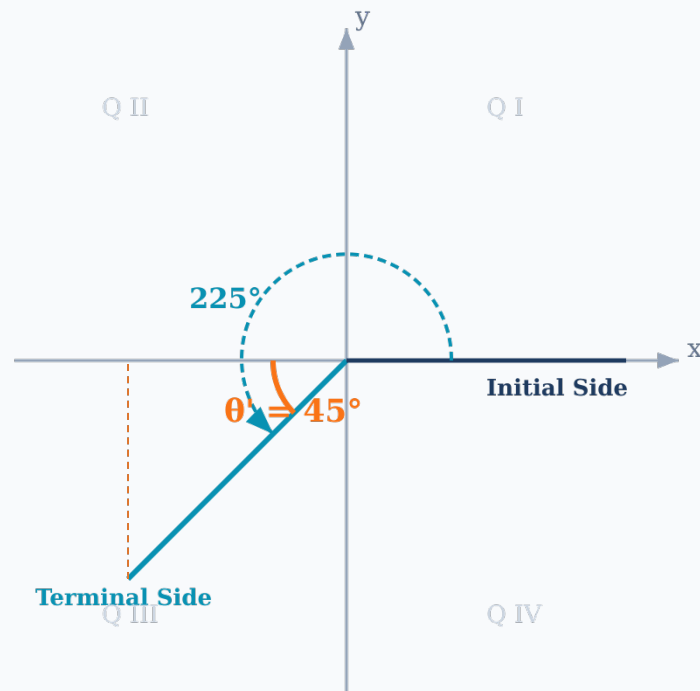
Reference Angle (θ'): The acute angle (0° to 90°) formed between the terminal side and the x-axis. Always measured to the nearest part of the x-axis.

Q I: $\theta' = \theta$

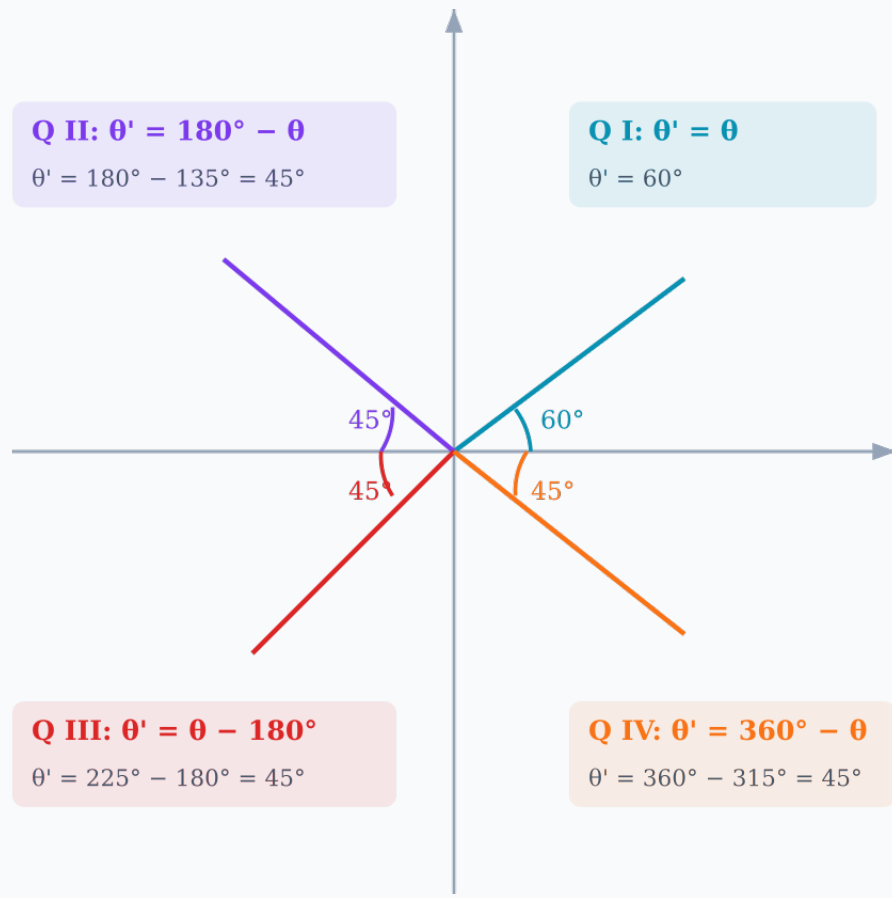
Q II: $\theta' = 180^\circ - \theta$ (or $\pi - \theta$)

Q III: $\theta' = \theta - 180^\circ$ (or $\theta - \pi$)

Q IV: $\theta' = 360^\circ - \theta$ (or $2\pi - \theta$)



Reference Angles — All Four Quadrants



Key Insight

The reference angle is always the same "core" acute angle — it tells you which special triangle values to use.

The quadrant determines the sign:

$$\text{trig}(\theta) = \pm \text{trig}(\theta')$$

Example: 225° has $\theta' = 45^\circ$

$$\sin 225^\circ = -\sin 45^\circ = -\sqrt{2}/2$$

(negative because $\sin < 0$ in Q III)

Reference Angles in Degrees — Type 1

Find the reference angle for: (a) 150° (b) 310° (c) 240°

(a) 150°

Q II

$$\theta' = 180^\circ - 150^\circ$$

$$\theta' = 30^\circ$$

(b) 310°

Q IV

$$\theta' = 360^\circ - 310^\circ$$

$$\theta' = 50^\circ$$

(c) 240°

Q III

$$\theta' = 240^\circ - 180^\circ$$

$$\theta' = 60^\circ$$

Reference Angles in Degrees — Type 2

Find the reference angle for: (a) -120° (b) 580° (c) -315°

Strategy: First find a coterminal angle between 0° and 360° , then apply the reference angle formula.

(a) -120°

$$-120^\circ + 360^\circ = 240^\circ$$

$$\text{Q III} \rightarrow \theta' = 240^\circ - 180^\circ$$

$$\theta' = 60^\circ$$

(b) 580°

$$580^\circ - 360^\circ = 220^\circ$$

$$\text{Q III} \rightarrow \theta' = 220^\circ - 180^\circ$$

$$\theta' = 40^\circ$$

(c) -315°

$$-315^\circ + 360^\circ = 45^\circ$$

$$\text{Q I} \rightarrow \theta' = \theta$$

$$\theta' = 45^\circ$$

Reference Angles in Radians

Find the reference angle: (a) $5\pi/6$ (b) $2\pi/3$ (c) $7\pi/4$ (d) $4\pi/3$

(a) $\theta = 5\pi/6$

Q II ($\approx 150^\circ$)

$$\theta' = \pi - 5\pi/6 = \pi/6$$

(b) $\theta = 2\pi/3$

Q II ($\approx 120^\circ$)

$$\theta' = \pi - 2\pi/3 = \pi/3$$

(c) $\theta = 7\pi/4$

Q IV ($\approx 315^\circ$)

$$\theta' = 2\pi - 7\pi/4 = \pi/4$$

(d) $\theta = 4\pi/3$

Q III ($\approx 240^\circ$)

$$\theta' = 4\pi/3 - \pi = \pi/3$$

Test Prep: Problem 2b asks for the reference angle of $\theta = 9\pi/7$. Since $9\pi/7 > \pi$ (Q III): $\theta' = 9\pi/7 - \pi = 9\pi/7 - 7\pi/7 = 2\pi/7$

Signs of Trig Functions — ASTC

S

Sine positive

$\sin + \cos - \tan -$

A

All positive

$\sin + \cos + \tan +$

T

Tangent positive

$\sin - \cos - \tan +$

C

Cosine positive

$\sin - \cos + \tan -$

"All Students Take Calculus"

Q I (A):

All 6 functions positive

Q II (S):

Only sin & csc positive

Q III (T):

Only tan & cot positive

Q IV (C):

Only cos & sec positive

Why? $\sin = y/r$, $\cos = x/r$. The sign depends on whether x and y are positive or negative in that quadrant.

Determining Quadrant from Trig Signs

Strategy: List which quadrants satisfy each condition, then find the intersection.

Given: $\tan \theta > 0$ and $\cos \theta < 0$

$\tan > 0$ in Q I, Q III

$\cos < 0$ in Q II, Q III

Answer: Quadrant III

Given: $\sin \theta > 0$ and $\sec \theta < 0$

$\sin > 0$ in Q I, Q II

$\sec < 0 \rightarrow \cos < 0$ in Q II, Q III

Answer: Quadrant II

Finding Trig Values from a Point

Terminal side of θ passes through $(-3, 4)$. Find all six trig values.

$$x = -3, y = 4$$

$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

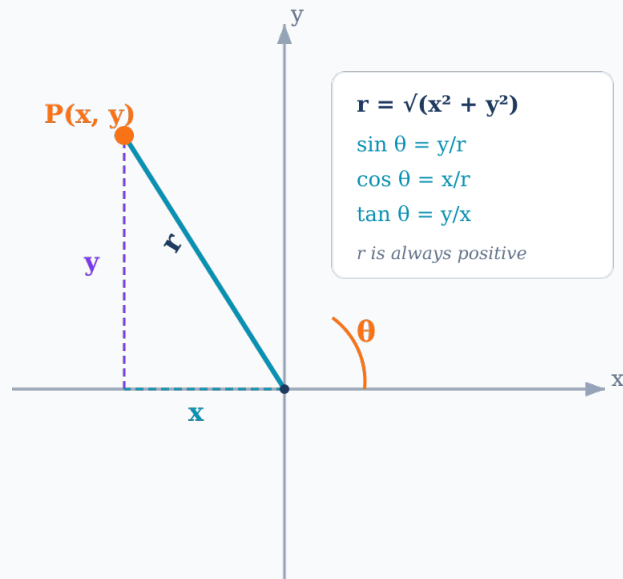
$$\sin \theta = 4/5$$

$$\cos \theta = -3/5$$

$$\tan \theta = -4/3$$

$$\csc \theta = 5/4 \quad \sec \theta = -5/3 \quad \cot \theta = -3/4$$

Point is in Q II $\rightarrow$ sin positive, cos & tan negative $\checkmark$



Test Prep — Trig Values from P(-2, -5)

Test 3, Problem 4b: Find the exact trig values of θ with terminal side through P(-2, -5).

Find r:

$x = -2, y = -5 \rightarrow$ Point is in Q III

$$r = \sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$$

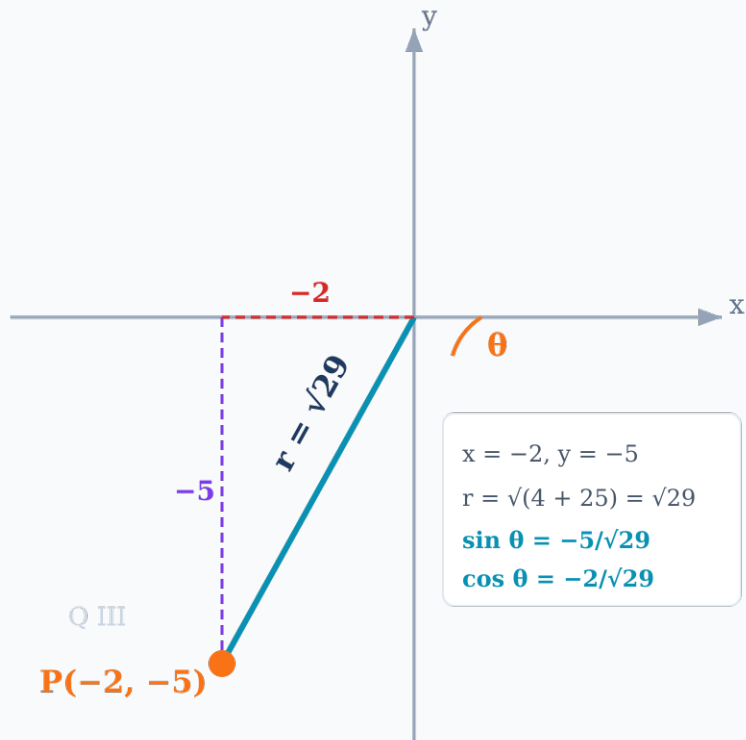
Answers:

$$\sin \theta = \frac{y}{r} = \frac{-5}{\sqrt{29}} = -\frac{5\sqrt{29}}{29}$$

$$\cos \theta = \frac{x}{r} = \frac{-2}{\sqrt{29}} = -\frac{2\sqrt{29}}{29}$$

$$\tan \theta = \frac{y}{x} = \frac{-5}{-2} = \frac{5}{2}$$

Q III check: $\sin -$, $\cos -$, $\tan +$ ✓



Finding Other Trig Values from One

Given $\tan \theta = -3/4$ and $\sin \theta < 0$, find $\sin \theta$ and $\cos \theta$.

Step 1: Determine the quadrant

$\tan < 0 \rightarrow \text{Q II or Q IV}$; $\sin < 0 \rightarrow \text{Q III or Q IV}$

Intersection $\rightarrow$ Quadrant IV ($x > 0, y < 0$)

Step 2: Set up the point

$\tan \theta = y/x = -3/4$ with Q IV signs $\rightarrow x = 4, y = -3$

$r = \sqrt{(16 + 9)} = \sqrt{25} = 5$

Step 3: Calculate

$\sin \theta = -3/5$ $\cos \theta = 4/5$

Exact Values at Common Angles

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	Ref
0°	0	1	0	—
$30^\circ (\pi/6)$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\pi/6$
$45^\circ (\pi/4)$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\pi/4$
$60^\circ (\pi/3)$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$\pi/3$
$90^\circ (\pi/2)$	1	0	undef	—

Use reference angle + quadrant sign to evaluate ANY multiple of 30° or 45°

Evaluating with Reference Angles

Find the exact value of: $\sin 240^\circ$ and $\cos 315^\circ$

$\sin 240^\circ$

$$\theta' = 240^\circ - 180^\circ = 60^\circ$$

Q III $\rightarrow$ sin is negative

$$\sin 240^\circ = -\sin 60^\circ$$

$$= -\sqrt{3}/2$$

$\cos 315^\circ$

$$\theta' = 360^\circ - 315^\circ = 45^\circ$$

Q IV $\rightarrow$ cos is positive

$$\cos 315^\circ = \cos 45^\circ$$

$$= \sqrt{2}/2$$

Test Prep — Angles from Trig Values

Test 3, Problem 3: Find two angles between 0° and 360° where $\sin \theta = \sqrt{3}/2$.

Step 1: Find the reference angle

$\sin \theta = \sqrt{3}/2 \rightarrow$ reference angle $= 60^\circ$

(from the 30-60-90 triangle: $\sin 60^\circ = \sqrt{3}/2$)

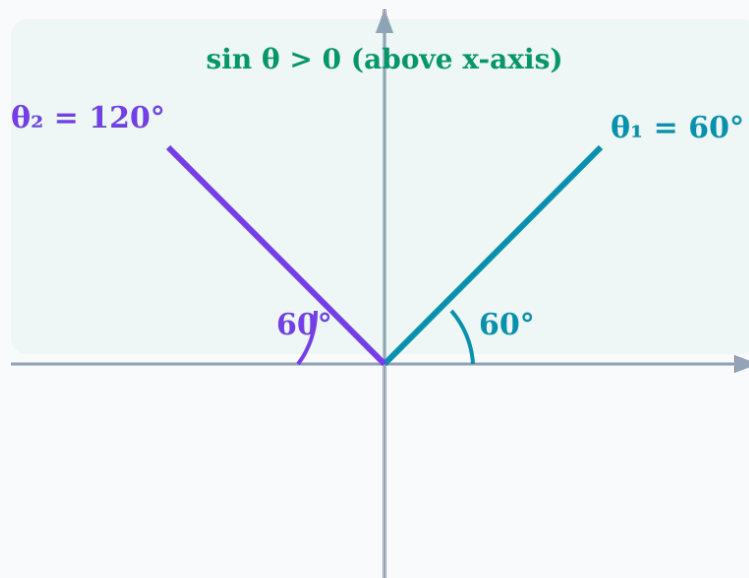
Step 2: Which quadrants?

$\sin > 0$ in Q I and Q II

Step 3: Apply formulas

Q I: $\theta = 60^\circ$

Q II: $\theta = 180^\circ - 60^\circ = 120^\circ$



$\sin \theta = \sqrt{3}/2 \rightarrow$ reference angle $= 60^\circ$

$\sin > 0$ in Q I and Q II $\rightarrow \theta = 60^\circ$ or $\theta = 120^\circ$

Area of a Triangle Using Trigonometry

$$\text{Area} = \frac{1}{2} \cdot b \cdot c \cdot \sin A$$

Two sides and their included angle

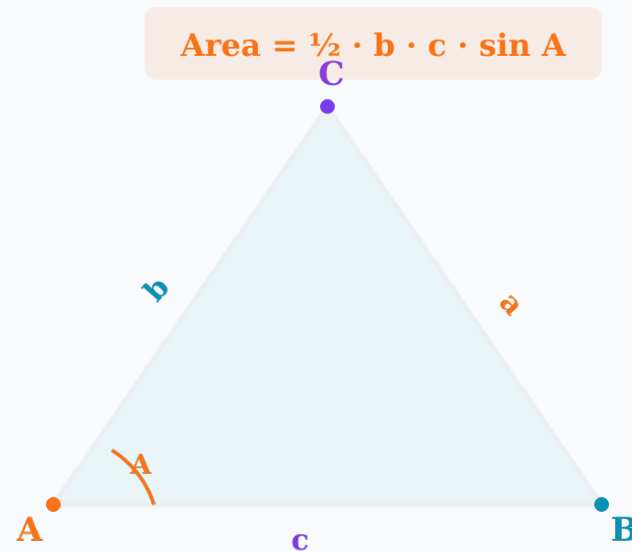
$$\text{Area} = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$

Equivalent with different labeling

$$\text{Area} = \frac{1}{2} \cdot a \cdot c \cdot \sin B$$

Equivalent with different labeling

Works for any triangle (not just right triangles) when you know two sides and the included angle. **This formula is on the Test 3 reference sheet.**



Test Prep — Triangle Area

Test 3, Problem 7: Find the area of a triangle (not a right triangle). $A = 76^\circ$, $b = 34$, $c = 21$.

Apply the formula:

$$\text{Area} = \frac{1}{2} \cdot b \cdot c \cdot \sin A$$

Substitute:

$$\text{Area} = \frac{1}{2} \times 34 \times 21 \times \sin 76^\circ$$

$$\text{Area} = \frac{1}{2} \times 714 \times 0.9703$$

$$\text{Area} = 357 \times 0.9703$$

$$\text{Area} \approx 346.4 \text{ square units}$$

Test Prep — Coterminal & Reference Angles

Test 3, Problem 2: Sketch, find positive & negative coterminal angles, and the reference angle.

(a) $\theta = -97^\circ$

Sketch: 97° clockwise $\rightarrow$ terminal side in Q III

Positive coterminal:

$$-97^\circ + 360^\circ = 263^\circ$$

Negative coterminal:

$$-97^\circ - 360^\circ = -457^\circ$$

Reference angle:

$$263^\circ \text{ is in Q III: } \theta' = 263^\circ - 180^\circ = 83^\circ$$

(b) $\theta = 9\pi/7$

Sketch: $9\pi/7 \approx 231.4^\circ \rightarrow$ Q III

Positive coterminal:

$$9\pi/7 + 2\pi = 9\pi/7 + 14\pi/7 = 23\pi/7$$

Negative coterminal:

$$9\pi/7 - 2\pi = 9\pi/7 - 14\pi/7 = -5\pi/7$$

Reference angle:

$$\text{Q III: } \theta' = 9\pi/7 - \pi = 9\pi/7 - 7\pi/7 = 2\pi/7$$

Module 11 Summary

8 ALEKS Topics | Section 5.3

Reference Angles

Always acute (0° – 90°), always measured to the x-axis. Find coterminal first if needed.

ASTC Signs

All Students Take Calculus — determines which trig functions are positive in each quadrant.

General Definitions

$\sin \theta = y/r$, $\cos \theta = x/r$, $\tan \theta = y/x$. Use point (x, y) on terminal side with $r = \sqrt{x^2 + y^2}$.

Key for Test 3

Problem 2 (coterminal/reference for -97° and $9\pi/7$), Problem 3 ($\sin \theta = \sqrt{3}/2$), Problem 4b (point $P(-2, -5)$), Problem 7 (area = $\frac{1}{2}bc \sin A$)

Next: Module 12 — Section 7.1 (Solving Right Triangles & Applications)