MATH 130

Geometry: Circles & Solids

Circumference, Area, Arc Length, Sectors, Radians, Velocity, and Solid Figures

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Learning Goals

By the end of this lesson, you will be able to:

Identify circle components

Name and define radius, diameter, chord, tangent, secant, arc, sector, and segment.

Calculate areas of basic shapes

Find areas of triangles, parallelograms, trapezoids, piecewise figures, and concentric circle regions.

Convert angles (DMS & radians)

Convert between DMS, decimal degrees, and radians.

Arc length, sectors & velocity

Apply $s = r\theta$, $A = \frac{1}{2}r^2\theta$, and $v = r\omega$ to solve problems.

Volume & surface area of solids

Use formulas for prisms, cylinders, cones, spheres, and pyramids.

SECTION 1

Course Readiness & Circle Basics

Area formulas for basic shapes, circumference, and circle terminology.

Area of Basic Shapes

Triangle, parallelogram, and trapezoid — the foundational area formulas you will use throughout this unit.

Triangle

$$A = \frac{1}{2} b h$$

Example: $b = 12$ cm, $h = 5$ cm
$A = \frac{1}{2}(12)(5) = 30 \text{ cm}^2$

Parallelogram

$$A = b h$$

Example: $b = 9$ in, $h = 4$ in
$A = (9)(4) = 36 \text{ in}^2$

Trapezoid

$$A = \frac{1}{2}(b_1 + b_2)h$$

Example: $b_1 = 8$, $b_2 = 14$, $h = 6$
$A = \frac{1}{2}(8+14)(6) = 66 \text{ units}^2$

Remember: h is always the perpendicular height, not a slanted side.

Piecewise Rectangular Figures

Break complex shapes into rectangles, find each area, then add them together.

Strategy

1. Identify how to split the figure into non-overlapping rectangles.

2. Calculate the area of each rectangle ($A = l \times w$).

3. Add all rectangle areas together for the total.

Tip: You can also find the area of the bounding rectangle and subtract the missing pieces.

Worked Example

An L-shaped room: overall 10 ft × 8 ft with a 4 ft × 3 ft piece removed from one corner.

Method 1 (subtraction):

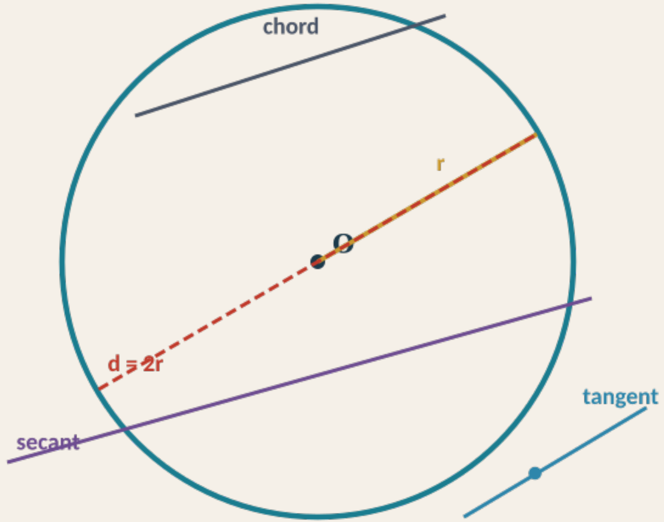
Full rectangle: $10 \times 8 = 80$ ft^2

Removed piece: $4 \times 3 = 12$ ft^2

Total area = $80 - 12 = 68$ ft^2

Circle Terminology

A circle is the set of all points in a plane equidistant from a fixed point (the center).



Radius

Center to circle. Half the diameter.

Diameter

Chord through center. Equal to 2r.

Chord

Segment with both endpoints on circle.

Tangent

Line touching circle at exactly one point.

Secant

Line intersecting circle at two points.

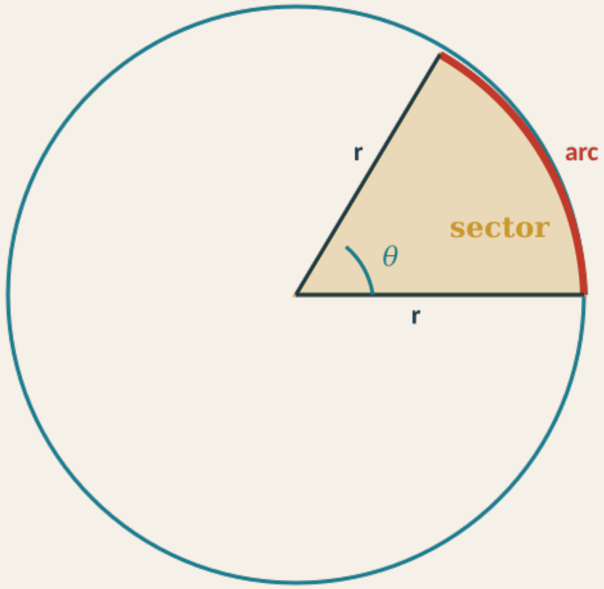
Center

Fixed equidistant point.

Arcs, Sectors & Segments

Key regions and curves defined within a circle.

Arc	A portion of the circumference. Minor arc < semicircle; major arc > semicircle.
Central Angle	Angle at the center formed by two radii. Its measure equals the intercepted arc.
Sector	The "pie-slice" region between two radii and the arc they intercept.
Segment	The region between a chord and its arc. A chord creates a minor and major segment.



$s = r\theta$ | $A = ½r^2\theta$

Circumference & Area

Two fundamental measurements of every circle.

Circumference

$$C = \pi d = 2\pi r$$

The distance around the circle.

Area

$$A = \pi r^2$$

The space enclosed by the circle. Always use the radius.

Example: Circumference

$r = 5 \rightarrow C = 2\pi(5) = 10\pi \approx 31.416$
$r = 7.6 \rightarrow C = 2\pi(7.6) = 15.2\pi \approx 47.752$

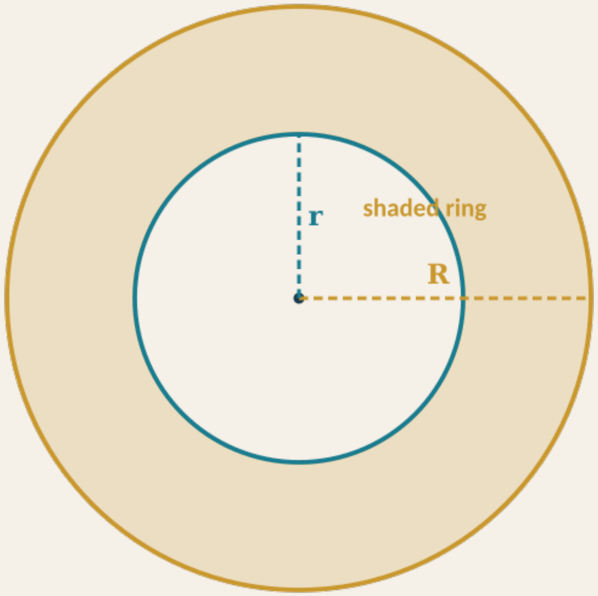
Example: Area (exact)

$r = 5 \rightarrow A = \pi(5^2) = 25\pi$
$r = 3 \rightarrow A = \pi(3^2) = 9\pi \approx 28.274$

Leave answers in terms of π for exact answers. Use $\pi \approx 3.14159$ for decimal approximations.

Area Between Concentric Circles

When one circle is inside another sharing the same center, the ring-shaped region between them is called an annulus.



$A = \pi(R^2 - r^2)$

Formula

$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$

R = outer radius, r = inner radius

Worked Example

A circular fountain has radius 4 ft. A walkway 3 ft wide surrounds it. Find the area of the walkway.

$R = 4 + 3 = 7$ ft, $r = 4$ ft

$A = \pi(7^2 - 4^2) = \pi(49 - 16) = 33\pi \approx 103.67 \text{ ft}^2$

Inscribed Angles

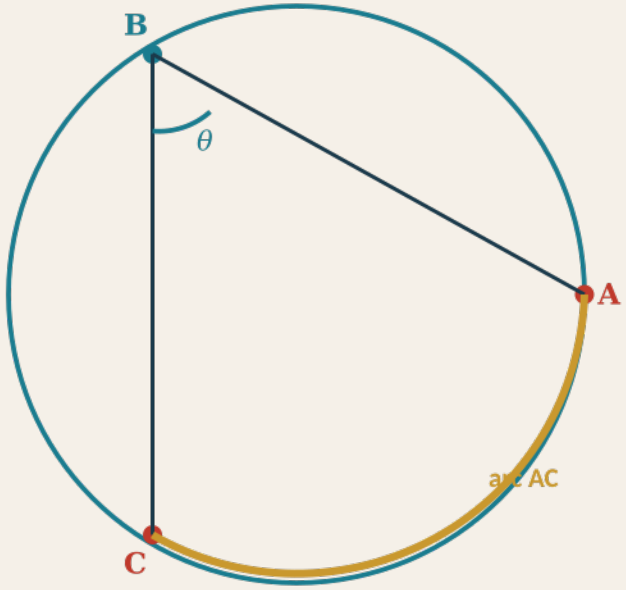
An inscribed angle has its vertex on the circle and its sides are chords.

Inscribed Angle Theorem

The measure of an inscribed angle is exactly one-half the measure of its intercepted arc.
$\angle ABC = \frac{1}{2} \cdot \text{arc } AC$

Corollary

An inscribed angle in a semicircle is always 90°, because the intercepted arc is 180°.



$\angle B = \frac{1}{2} \cdot \text{arc } AC$

Chord & Angle Theorems

When two chords intersect inside a circle, the angles formed relate to the intercepted arcs.

Intersecting Chords Angle

$$\angle = \frac{1}{2}(arc_1 + arc_2)$$

The angle equals half the sum of the two intercepted arcs.

Vertical & Supplementary

Vertical angles are congruent.
Adjacent angles are supplementary (sum = 180°).

Inscribed Angle ↔ Arc

Inscribed angle → arc = 2 × angle
Arc → inscribed angle = ½ × arc

Diameter = 180° Arc

A diameter divides the circle into two semicircles of 180° each.

Worked Example: Circle Angles

∠AED = 133° and arc AB = 52°. AC is a diameter (BD is not). Find ∠AEB, ∠ACB, and arc CD.

Part (a): Find ∠AEB

∠AEB and ∠AED are supplementary (straight line).

∠AEB = 180° − 133° = 47°

Part (b): Find ∠ACB

∠ACB is inscribed, intercepting arc AB = 52°.

∠ACB = ½ × 52° = 26°

Part (c): Find arc CD

133° = ½(arc AD + arc BC). Arc BC = 180° − 52° = 128°

266° = arc AD + 128° → arc AD = 138°

arc CD = 180° − 138° = 42°

(a) ∠AEB

= 47°

(b) ∠ACB

= 26°

(c) arc CD

= 42°

SECTION 2

Angle Measure & Arc Length

DMS notation, radians, degree-radian conversion, arc length, and sector area formulas.

Degrees, Minutes & Seconds

Angles in navigation and surveying use DMS notation. 1° = 60′ (minutes), 1′ = 60″ (seconds).

DMS → Decimal Degrees

$$\text{Decimal} = D + M/60 + S/3600$$

Divide minutes by 60, seconds by 3600, then add all parts.

Decimal → DMS

1. Whole part = degrees
2. Decimal × 60 → whole part = minutes
3. Remaining decimal × 60 = seconds

Example: 37°41′11″ → Decimal

$= 37 + 41/60 + 11/3600$
$= 37 + 0.6833 + 0.003056$
$= 37.6864°$

Example: 36.3492° → DMS

D = 36°. 0.3492 × 60 = 20.952 → M = 20′
0.952 × 60 = 57.12 → S ≈ 57″
Result: 36°20′57″

Degrees & Radians

If a central angle intercepts an arc equal in length to the radius, that angle is exactly 1 radian.

Key Relationship

$$2\pi \text{ rad} = 360°$$
$$\pi \text{ rad} = 180°$$

Degrees → Radians

Multiply by π
Divide by 180

Radians → Degrees

Multiply by 180
Divide by π

One radian ≈ 57.296°. Think of it as the angle that "unwraps" one radius-length of arc.

Conversion Examples

Practice converting between degrees and radians in both directions.

Radians → Degrees

$\pi/6 \to (\pi/6)(180/\pi) = 30°$

$2.7 \text{ rad} \to (2.7)(180/\pi) \approx 154.699°$

$1 \text{ rad} \to (1)(180/\pi) \approx 57.296°$

Degrees → Radians

$45° \to (45)(\pi/180) = \pi/4$

$120° \to (120)(\pi/180) = 2\pi/3$

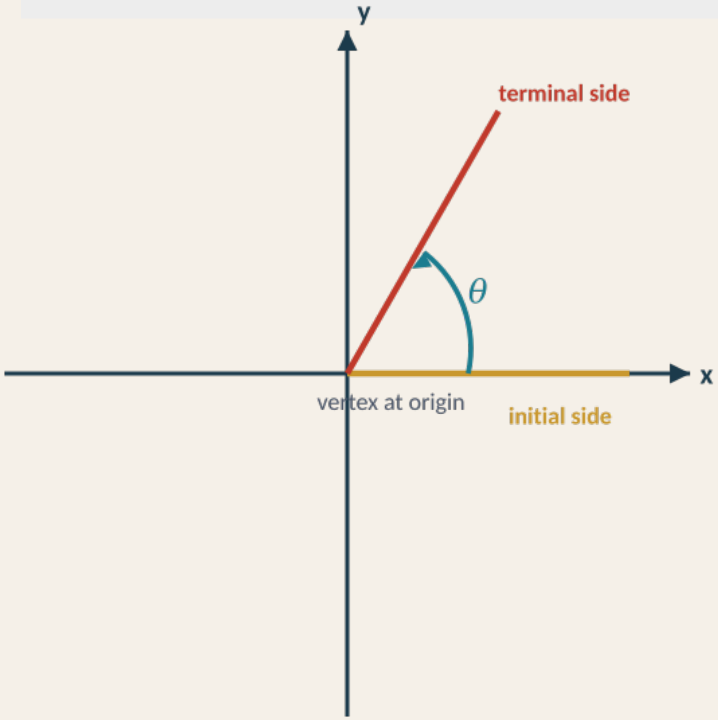
$255° \to (255)(\pi/180) = 17\pi/12$

$1° \to \pi/180 \approx 0.01745 \text{ rad}$

Remember: π rad = 180° is the bridge. Multiply by whichever fraction cancels the unit you want to eliminate.

Sketching Angles in Standard Position

Standard position: vertex at origin, initial side along the positive x-axis.



How to Sketch

1. Place the initial side along the positive x-axis.
2. Rotate counterclockwise for positive angles, clockwise for negative angles.
3. Draw the terminal side at the given angle.

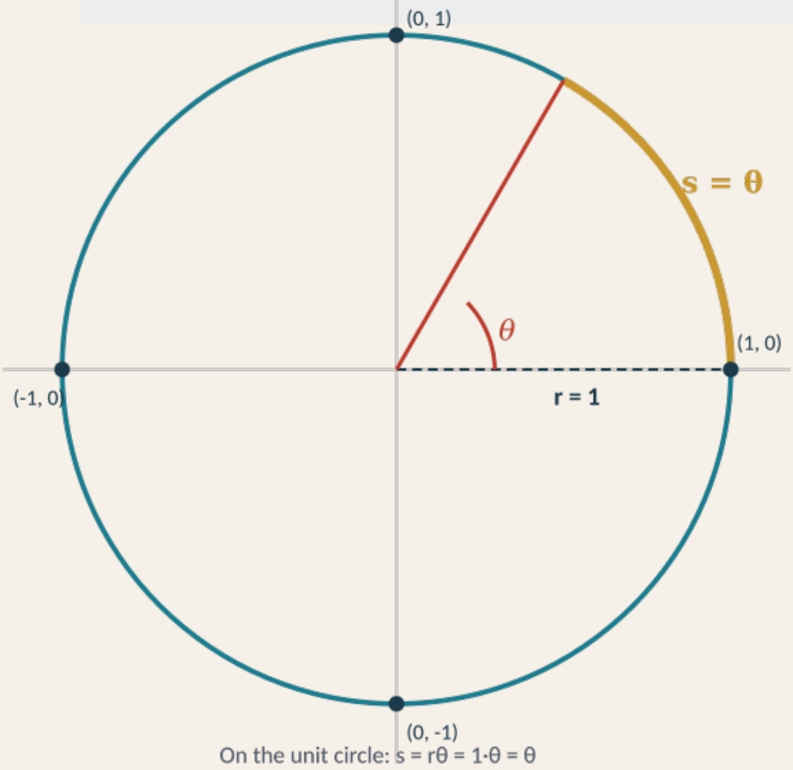
Key Angles to Know

$\pi/6 = 30°$ $\pi/4 = 45°$ $\pi/3 = 60°$
$\pi/2 = 90°$ $\pi = 180°$ $3\pi/2 = 270°$
$2\pi = 360°$

Negative: $-\pi/4$ is 45° clockwise from x-axis.

Arc Length on the Unit Circle

On the unit circle ($r = 1$), the arc length equals the central angle in radians: $s = \vartheta$.



Why $s = \theta$ on the Unit Circle

Since $s = r\theta$ and $r = 1$:
$s = (1)\theta = \theta$

The radian measure of an angle IS the arc length on the unit circle.

Example

Central angle = $2\pi/3$. On the unit circle:
Arc length $s = 2\pi/3 \approx 2.094$

On a circle with $r = 5$:
$s = 5 \cdot (2\pi/3) = 10\pi/3 \approx 10.472$

Arc Length

The distance along the curved part of a circle between two points. ϑ must be in radians.

$s = r\theta$

s = arc length, r = radius, θ = central angle (radians)

If θ is in degrees:

Convert to radians first by multiplying by $\pi/180$, then apply $s = r\theta$.

Example 1

Central angle = $\pi/3$, radius = 10
$s = r\theta = 10 \cdot (\pi/3) = 10\pi/3 \approx 10.472$

Example 2

Central angle = $45°$, radius = 21
Convert: $45° \times \pi/180 = \pi/4$
$s = 21 \cdot (\pi/4) = 21\pi/4 \approx 16.493$

Area of a Sector

A sector is the "pie-slice" region between two radii and an arc. Its area is proportional to ϑ.

$$A = \tfrac{1}{2}r^2\theta$$

A = area, r = radius, θ = central angle (radians)

Alternatively (degrees):

$A = (\theta/360) \cdot \pi r^2$
This is simply the fraction of the full circle's area.

Example 1

Central angle = $\pi/3$, radius = 10
$A = \tfrac{1}{2}(10^2)(\pi/3) = 50\pi/3 \approx 52.360$

Example 2

Central angle = 45°, radius = 21
Convert: $45° \times \pi/180 = \pi/4$
$A = \tfrac{1}{2}(21^2)(\pi/4) = 441\pi/8 \approx 173.18$

Real-World Sector Problems

Applying arc length and sector area to aviation and everyday scenarios.

Aviation Arc

Arc south on 12 NM arc from 090 to 210 radial.

Central angle: $210° - 90° = 120° = 2\pi/3$ rad
Arc length: $s = 12(2\pi/3) = 8\pi \approx 25.13$ NM
Time at 120 kt: $25.13/120 \approx 0.209$ hr ≈ 12.6 min

TLAR check: 120° is ⅓ of 360°, so $\frac{1}{3} \times 2\pi(12) \approx 25$ NM ✓

Sprinkler Coverage

A sprinkler sprays water 20 ft, covering a 150° sector. Find the area of the watered region.

Convert: $150° \times \pi/180 = 5\pi/6$ rad
Area: $A = \frac{1}{2}(20^2)(5\pi/6)$
$= \frac{1}{2}(400)(5\pi/6)$
$= 1000\pi/3 \approx 1047.2$ ft^2

That is about $150/360 \approx 41.7\%$ of the full circle.

SECTION 3

Linear & Angular Velocity

Connecting rotational speed to the speed of a point on a spinning object.

Linear & Angular Velocity

Two ways to describe how fast something spins — and how they connect.

Angular Velocity (ω)

$$\omega = \theta / t$$

How fast the angle changes. Measured in rad/s or rad/min.

Linear Velocity (v)

$$v = r\omega = s / t$$

Speed of a point on the rotating object. Farther from center → faster.

Converting RPM to Radians

1 revolution = 2π radians. So n RPM = $n \times 2\pi$ rad/min. Always convert before using $v = r\omega$.

Key insight: $v = r\omega$ — linear velocity is directly proportional to both radius and angular velocity.

Worked Example: Riverboat Paddles

Radius = 8.50 ft, turning at 20.0 RPM. Find the speed of a paddle tip.

Solution

Step 1 — Convert RPM to angular velocity:

$\omega = 20.0$ rev/min $\times 2\pi$ rad/rev $= 40\pi$ rad/min

Step 2 — Apply $v = r\omega$:

$v = 8.50$ ft $\times 40\pi$ rad/min $= 340\pi$ ft/min

Step 3 — Convert to convenient units:

$v = 340\pi \approx 1068$ ft/min ≈ 12.14 mi/hr

Answer

≈ 1068 ft/min

≈ 12.14 mi/hr

The paddle tip moves at about 12 miles per hour through the water.

More Velocity Examples

Drive shaft rotation and wind turbine blade tips.

Drive Shaft Rotation

Tachometer reads 3200 RPM. Angle in 1 second?

$\omega = 3200 \times 2\pi/60 \approx 335.1$ rad/s

In 1 s: $\theta = 335.1$ rad ≈ 53.3 revolutions

Wind Turbine Blade Tips

Blades 116 ft long at 22.0 RPM. Tip speed?

$\omega = 22.0 \times 2\pi = 44\pi$ rad/min

$v = 116 \times 44\pi \approx 16{,}035$ ft/min

≈ 182.2 mph

Bonus: Turbine Sweep Area

$A = \pi r^2 = \pi(116)^2 = 13{,}456\pi \approx 42{,}273\ \text{ft}^2 \div 43{,}560\ \text{ft}^2/\text{acre} \approx 0.97$ acres — nearly a full acre of wind capture.

SECTION 4

Solid Geometric Figures

Volume and surface area formulas for common three-dimensional shapes.

Volume & Surface Area

B = base area, P = perimeter, l = slant height, h = height, r = radius.

Rectangular Prism

$$V = lwh$$

Surface Area: $SA = 2(lw+lh+wh)$

Cylinder

$$V = \pi r^2 h$$

Surface Area: $SA = 2\pi r^2+2\pi rh$

Cone

$$V = \tfrac{1}{3}\pi r^2 h$$

Surface Area: $SA = \pi r^2+\pi rl$

Sphere

$$V = \tfrac{4}{3}\pi r^3$$

Surface Area: $SA = 4\pi r^2$

Right Pyramid

$$V = \tfrac{1}{3}Bh$$

Surface Area: $SA = B+\tfrac{1}{2}Pl$

Triangular Prism

$$V = Bh$$

Surface Area: $SA = 2B+Ph$

Worked Example: Drain Pipe

A 12 ft pipe (inner diameter 4 in) is clogged. Will two 5-gallon buckets hold the water? (1 gal = 231 in^3)

Step 1 — Convert to consistent units:
Length: 12 ft = 144 in. Radius: 4/2 = 2 in.

Step 2 — Volume of the cylinder:
$V = \pi r^2 h = \pi(2^2)(144) = 576\pi \approx 1809.56$ in^3

Step 3 — Convert to gallons:
$V = 1809.56 \div 231 \approx 7.83$ gallons

Step 4 — Compare:
Two 5-gal buckets = 10 gal. Since 7.83 < 10, yes!

Answer

Yes ✓

≈ 7.83 gal

Two 5-gallon buckets provide more than enough capacity (10 gal > 7.83 gal).

Formula Quick Reference

All the key formulas from this unit in one place.

Circle & Area

$C = 2\pi r = \pi d$ $A = \pi r^2$

Annulus: $A = \pi(R^2 - r^2)$

Triangle: $A = \frac{1}{2}bh$

Parallelogram: $A = bh$

Trapezoid: $A = \frac{1}{2}(b_1+b_2)h$

Arc Length & Sector

$s = r\theta$ $A = \frac{1}{2}r^2\theta$ (θ in radians)

Conversions

Deg → Rad: multiply by $\pi/180$

Rad → Deg: multiply by $180/\pi$

DMS → Dec: $D + M/60 + S/3600$

Velocity

$\omega = \theta/t$ $v = r\omega = s/t$

RPM → rad/min: $n \times 2\pi$

Solids

Rect. Prism: $V=lwh$ $SA=2(lw+lh+wh)$

Cylinder: $V=\pi r^2 h$ $SA=2\pi r^2+2\pi rh$

Cone: $V=\frac{1}{3}\pi r^2 h$ $SA=\pi r^2+\pi rl$

Sphere: $V=\frac{4}{3}\pi r^3$ $SA=4\pi r^2$

Pyramid: $V=\frac{1}{3}Bh$ $SA=B+\frac{1}{2}Pl$

Inscribed Angle

$\angle$inscribed = $\frac{1}{2}$ × intercepted arc

Watch out: Always convert to radians before using $s = r\vartheta$ or $A = \frac{1}{2}r^2\vartheta$. Always use radius (not diameter) for area.

References

Katz, V. J. (1998). A History of Mathematics: An Introduction (2nd ed.). New York, NY: Addison-Wesley.