Section 4.3

log

# Logarithmic Functions

College Algebra & Trigonometry

*Lecture Notes with Worked Examples*

# LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Convert between logarithmic and exponential forms
- Evaluate logarithmic expressions (including common and natural logs)
- Apply basic properties of logarithms
- Graph logarithmic functions and identify key features
- Use the change-of-base formula
- Solve simple logarithmic equations

# REVIEW: INVERSE FUNCTIONS

Before we define logarithms, recall the concept of an inverse function:

## The Big Idea

The inverse of a function undoes what the function does. If $f$ assigns $y$ to $x$, then $f^{-1}$ assigns $x$ back to $y$.

## Example

If $f(3) = 5$, then $f^{-1}(5) = 3$

## Verifying Inverses

$g(f(x)) = x$ for all $x$ in domain of $f$

$f(g(x)) = x$ for all $x$ in domain of $g$

## Why This Matters for Logarithms

The exponential function $f(x) = b^x$ is one-to-one, so it has an inverse: $f^{-1}(x) = \log_b(x)$.

# DEFINITION

## Logarithmic Function

If $x$ and $b$ are positive real numbers such that $b \neq 1$, then the logarithmic function with base $b$ is:

$$y = log_b(x) \text{ is equivalent to } b^y = x$$

## Logarithmic vs. Exponential

| Logarithmic Form | | Exponential Form |
|---|---|---|
| $y = log_b(x)$ | $\Leftrightarrow$ | $b^y = x$ |
| *"What power of b gives x?"* | | *"b raised to y equals x"* |

# TRY IT: CONVERT TO EXPONENTIAL FORM

Write each logarithmic equation in exponential form.

a) $\log_4(64) = 3 \quad \rightarrow \quad ?$

b) $\log_3(1/9) = -2 \quad \rightarrow \quad ?$

c) $\log_{10}(1000) = 3 \quad \rightarrow \quad ?$

d) $\ln(e^5) = 5 \quad \rightarrow \quad ?$

*Pause and try this before advancing!*

# SOLUTION: CONVERT TO EXPONENTIAL FORM

**a)** $\log_4(64) = 3$ → $4^3 = 64$

*4 cubed is 64 ✓*

**b)** $\log_3(1/9) = -2$ → $3^{-2} = 1/9$

*$3^{-2} = 1/9$ ✓*

**c)** $\log_{10}(1000) = 3$ → $10^3 = 1000$

*10 cubed is 1000 ✓*

**d)** $\ln(e^5) = 5$ → $e^5 = e^5$

*ln means log base e*

**Remember:** $\log_b(x) = y$ means $b^y = x$. The logarithm is always the exponent!

# TRY IT: CONVERT TO LOGARITHMIC FORM

Write each exponential equation in logarithmic form.

a) $5^3 = 125 \quad \rightarrow \quad ?$

b) $4^{-3} = 1/64 \quad \rightarrow \quad ?$

c) $10^0 = 1 \quad \rightarrow \quad ?$

d) $e^1 = e \quad \rightarrow \quad ?$

*Pause and try this before advancing!*

# SOLUTION: CONVERT TO LOGARITHMIC FORM

a) $5^3 = 125$ $\rightarrow$ $\log_5(125) = 3$

b) $4^{-3} = 1/64$ $\rightarrow$ $\log_4(1/64) = -3$

c) $10^0 = 1$ $\rightarrow$ $\log(1) = 0$

d) $e^1 = e$ $\rightarrow$ $\ln(e) = 1$

As a function: $f(x) = \log_4(x)$ For example, $f(1/64) = -3$ because $4^{-3} = 1/64$.

# EVALUATING LOGARITHMIC EXPRESSIONS

To evaluate $\log_b(x)$, ask: "$b$ raised to what power gives $x$?"

### Example 1

Evaluate $\log_2(32)$

Ask: 2 raised to what power = 32?

$2^5 = 32$, so $\log_2(32) = \mathbf{5}$

### Example 2

Evaluate $\log_3(1/27)$

Ask: 3 raised to what power = 1/27?

$3^{-3} = 1/27$, so $\log_3(1/27) = \mathbf{-3}$

### Example 3

Evaluate $\log_5(\sqrt{5})$

$\sqrt{5} = 5^{1/2}$, so $\log_5(5^{1/2}) = \mathbf{1/2}$

### Example 4

Evaluate $\log_9(1)$

Ask: 9 raised to what power = 1?

$9^0 = 1$, so $\log_9(1) = \mathbf{0}$

# COMMON AND NATURAL LOGARITHMS

## Common Logarithm

$$\log_{10}(x) = \log(x)$$

Base 10 is the default — when you see "log" with no base written, it means base 10.

Your calculator's LOG button is base 10.

## Natural Logarithm

$$\log_e(x) = \ln(x)$$

Base e ≈ 2.71828 — the natural base from Section 4.2. Used extensively in calculus.

Your calculator's LN button is base e.

Calculator: $\log(140) \approx 2.1461 \rightarrow$ Check: $10^{2.1461} \approx 140$ ✓ $\ln(60) \approx 4.0943 \rightarrow$ Check: $e^{4.0943} \approx 60$ ✓

# TRY IT: EVALUATE WITH A CALCULATOR

Use your calculator. Round to 4 decimal places. Check using exponential form.

a) $\log(250) = ?$

b) $\ln(40) = ?$

c) $\log(0.015) = ?$

d) $\ln(1) = ?$

*Pause and try this before advancing!*

# SOLUTION: EVALUATE WITH A CALCULATOR

**a)** $\log(250) \approx 2.3979$

Check: $10^{2.3979} \approx 250$ ✓

**b)** $\ln(40) \approx 3.6889$

Check: $e^{3.6889} \approx 40$ ✓

**c)** $\log(0.015) \approx -1.8239$

Check: $10^{-1.8239} \approx 0.015$ ✓ (Negative because $0.015 < 1$)

**d)** $\ln(1) = 0$

Check: $e^{0} = 1$ ✓ (No calculator needed! $\log_b(1) = 0$ for any base)

# BASIC PROPERTIES OF LOGARITHMS

$$\log_b(1) = 0$$

because $b^0 = 1$

$\log(1) = 0$

$\ln(1) = 0$

$$\log_b(b) = 1$$

because $b^1 = b$

$\log(10) = 1$

$\ln(e) = 1$

$$\log_b(b^x) = x$$

log undoes exponent

$\log(10^x) = x$

$\ln(e^x) = x$

**Inverse Property**

$$b^{\log\_b(x)} = x$$

$10^{\log x} = x$ $\quad$ $e^{\ln x} = x$

**One-to-One Property**

If $\log_b(x) = \log_b(y)$

then $x = y$

Equal logs (same base) → equal arguments

*These properties come directly from the definition. They are used constantly in solving equations!*

# TRY IT: SIMPLIFY USING PROPERTIES

Simplify each expression without a calculator.

a) $\log_7(1)$

b) $\ln(e^{-5/2})$

c) $\log(10^8)$

d) $5^{\log_5(12)}$

e) $\log_4(4^3)$

f) $e^{\ln 7}$

*Pause and try this before advancing!*

# SOLUTION: SIMPLIFY USING PROPERTIES

a) $\log_7(1) = 0$

because $7^0 = 1$

d) $5^{\log_5(12)} = 12$

because $b^{\log\_b(x)} = x$

b) $\ln(e^{-5/2}) = -5/2$

because $\ln(e^x) = x$

e) $\log_4(4^3) = 3$

because $\log_b(b^x) = x$

c) $\log(10^8) = 8$

because $\log(10^x) = x$

f) $e^{\ln 7} = 7$

because $e^{\ln x} = x$

All follow directly from the basic properties.  No calculator needed!

# GRAPHS OF LOGARITHMIC FUNCTIONS

Since $f(x) = \log_b(x)$ is the inverse of $g(x) = b^x$, their graphs are reflections across $y = x$.

## Graphing $y = \log_2(x)$

Rewrite as $2^y = x$ and choose $y$-values:

| y | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $x = 2^y$ | ¼ | ½ | 1 | 2 | 4 | 8 |

Points: (¼,−2), (½,−1), (1,0), (2,1), (4,2), (8,3)

*Tip: Choose y-values first, then compute $x = 2^y$. Avoids fractions!*

## $f(x) = 2^x$ (exponential)

Horizontal asymptote: $y = 0$
$y$-intercept: $(0, 1)$
Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

## $f(x) = \log_2(x)$ (logarithmic)

Vertical asymptote: $x = 0$
$x$-intercept: $(1, 0)$
Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

**Inverse functions swap domain/range, x/y-intercepts, and horizontal/vertical asymptotes!**

# PROPERTIES OF $f(x) = \log_b(x)$

Both cases: Domain: $(0, \infty)$ Range: $(-\infty, \infty)$ $x$-intercept: $(1, 0)$ Asymptote: $x = 0$

## $b > 1$ (e.g., $\log_2 x$)

Increasing function
As
$x \to 0^+, f(x) \to -\infty$
As
$x \to \infty, f(x) \to \infty$

Graph rises slowly to the right
Passes through $(1, 0)$ and (
$b$, 1)

## $0 < b < 1$ (e.g., $\log_{1/2} x$)

Decreasing function
As
$x \to 0^+, f(x) \to \infty$
As
$x \to \infty, f(x) \to -\infty$

Graph falls slowly to the right
Passes through $(1, 0)$ and (
$b$, 1)

*Key Question: How do you tell growth from decay? Look at the base b, same as with exponentials!*

# TRY IT: GRAPH A LOGARITHMIC FUNCTION

Graph $f(x) = \log_2(x + 1) + 2$. Identify the domain, range, and vertical asymptote.

*Hint: Start with the parent function $y = \log_2(x)$. How does it shift?*

## Transformation Reference

| | |
|---|---|
| $f(x) = \log_b(x - h) + k$ | Shifts right h, up k |
| $f(x) = \log_b(x + h)$ | Shifts LEFT h units |
| $f(x) = \log_b(x) + k$ | Shifts UP k units |
| $f(x) = -\log_b(x)$ | Reflects over x-axis |

*Pause and try this before advancing!*

# SOLUTION: GRAPH $f(x) = \log_2(x+1) + 2$

## Transformations Applied

Parent: $y = \log_2(x)$
Shift left 1: $(x+1)$ moves asymptote to $x = -1$
Shift up 2: $+2$ moves $x$-intercept up

## Key Features

Vertical asymptote: $x = -1$
Domain: $(-1, \infty)$
Range: $(-\infty, \infty)$

| x | −¾ | 0 | 1 | 3 | 7 |
|---|---|---|---|---|---|
| f(x) | 0 | 2 | 3 | 4 | 5 |

## Step-by-Step

$f(0) = \log_2(0+1) + 2 = \log_2(1) + 2 = 0 + 2 = 2$ ($y$-intercept at $(0, 2)$)
$f(1) = \log_2(1+1) + 2 = \log_2(2) + 2 = 1 + 2 = 3$
$f(3) = \log_2(3+1) + 2 = \log_2(4) + 2 = 2 + 2 = 4$
$f(7) = \log_2(7+1) + 2 = \log_2(8) + 2 = 3 + 2 = 5$

# DOMAIN OF LOGARITHMIC FUNCTIONS

## Critical Rule

The argument of a logarithm must be strictly positive. For $f(x) = \log_b(g(x))$, the domain requires $g(x) > 0$.

### $f(x) = \log(3x - 6)$

Set argument > 0:

3x − 6 > 0 → 3x > 6 → x > 2

Domain: (2, ∞)

Vertical asymptote: x = 2

### $f(x) = \ln(x^2 + 4)$

Set argument $> 0$:

$x^2 + 4 > 0 \rightarrow$ always true!

(

$x^2 \geq 0$, so $x^2 + 4 \geq 4 > 0$)

Domain: $(-\infty, \infty)$

### $f(x) = \log_2(x^2 - 4)$

Set: $x^2 - 4 > 0 \rightarrow (x-2)(x+2) > 0 \rightarrow x < -2$ or $x > 2$ Domain: $(-\infty, -2) \cup (2, \infty)$

*Find boundary points, test intervals. Vertical asymptotes at x = −2 and x = 2.*

# SOLVING LOGARITHMIC EQUATIONS

Strategy: Convert to exponential form or use the one-to-one property.

## Method 1: Convert to Exponential

Solve: $\log_2(x - 5) = 4$
Convert: $x - 5 = 2^4 = 16$
Solve:
$x = 16 + 5 = \mathbf{21}$

Check: $\log_2(21 - 5) = \log_2(16) = 4$ ✓

## Method 2: One-to-One Property

Solve: $\log_5(x^2 + 4) = \log_5(3x + 14)$
Drop logs: $x^2 + 4 = 3x + 14$
$x^2 - 3x - 10 = 0$
$(x - 5)(x + 2) = 0$
$x = 5$ or $x = -2$ Both check ✓

## Method 3: Using Inverse Properties

Solve: $e^{\ln 3x} = 0.2$

Simplify: $3x = 0.2$ (since $e^{\ln u} = u$)
$x = 0.2/3 \approx \mathbf{0.067}$

Solve: $10^{2x-1} = 145$

Take log: $2x - 1 = \log(145)$
$x = (\log(145) + 1)/2 \approx \mathbf{1.581}$

# TRY IT: SOLVE LOGARITHMIC EQUATIONS

Solve each equation.

a) $\ln(x^2 - x) = \ln 6$

b) $e^{1-4t} = 12.405$ (solve for $t$)

c) $\log_x(232) = 21$ (solve for $x$, round to 3 dec.)

d) $3^x = 175$ (round to 3 decimal places)

*Pause and try this before advancing!*

# SOLUTION: SOLVE LOGARITHMIC EQUATIONS

### a) $\ln(x^2 - x) = \ln 6$

One-to-one: $x^2 - x = 6$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$ Both check ✓

### b) $e^{1-4t} = 12.405$

Take ln: $1 - 4t = \ln(12.405)$

$-4t = \ln(12.405) - 1$

$t = (\ln(12.405) - 1)/(-4)$

$t \approx$ **$-0.380$**

### c) $\log_x(232) = 21$

Exponential form: $x^{21} = 232$

$x = 232^{1/21}$

$x \approx$ **1.295**

### d) $3^x = 175$

Take log: $x \cdot \log(3) = \log(175)$

$x = \log(175)/\log(3)$

$x \approx$ **4.700**

**Always check your answers!**

Verify all log arguments remain positive. Extraneous solutions can arise with logarithms.

# ⇄ CHANGE-OF-BASE FORMULA

Your calculator only has LOG (base 10) and LN (base e). To evaluate other bases, convert!

$$\log_a(x) = \log(x) \,/\, \log(a) = \ln(x) \,/\, \ln(a)$$

*"To convert from base a, divide by log(a)."*

**Example: Evaluate $\log_4(25)$**

**Using LOG:**

$\log_4(25) = \log(25) \,/\, \log(4) = 1.39794 \,/\, 0.60206 \approx \mathbf{2.322}$

**Using LN:**

$\log_4(25) = \ln(25) \,/\, \ln(4) = 3.21888 \,/\, 1.38629 \approx \mathbf{2.322}$ (same answer!)

# TRY IT: CHANGE-OF-BASE FORMULA

Evaluate using change-of-base. Round to 3 decimal places.

a) $\log_3(50)$

b) $\log_7(0.3)$

c) $\log_2(100)$

*Pause and try this before advancing!*

# SOLUTION: CHANGE-OF-BASE FORMULA

**a)** $\log_3(50) = \log(50)/\log(3)$

$= 1.69897 / 0.47712 \approx 3.561$

Check: $3^{3.561} \approx 50$ ✓

**b)** $\log_7(0.3) = \log(0.3)/\log(7)$

$= -0.52288 / 0.84510 \approx -0.619$

Check: $7^{-0.619} \approx 0.3$ ✓ (Negative because $0.3 < 1$)

**c)** $\log_2(100) = \log(100)/\log(2)$

$= 2 / 0.30103 \approx 6.644$

Check: $2^{6.644} \approx 100$ ✓ ($2^6 = 64$, $2^7 = 128$)

# APPLICATION: STAR MAGNITUDE

Astronomers use logarithmic scales to measure star brightness.

$$\mathbf{M} = \mathbf{m} - 5 \cdot \log(\mathbf{d}/10)$$

**M** = absolute magnitude **m** = apparent magnitude **d** = distance in parsecs

## Sirius (Dog Star)

$m = -1.44,\ d = 2.637$ parsecs

$M = -1.44 - 5 \cdot \log(2.637/10)$
$= -1.44 - 5 \cdot \log(0.2637)$
$= -1.44 - 5(-0.5788)$
$= -1.44 + 2.894$
$\approx \mathbf{1.45}$

## Our Sun

$m = -26.74,\ d = 4.848 \times 10^{-6}$ pc

$M = -26.74 - 5 \cdot \log(4.848 \times 10^{-6}/10)$
$= -26.74 - 5 \cdot \log(4.848 \times 10^{-7})$
$= -26.74 - 5(-6.3143)$
$= -26.74 + 31.572$
$\approx \mathbf{4.83}$

# COMMON MISTAKES TO AVOID

✗ **Thinking $\log_b(x + y) = \log_b(x) + \log_b(y)$**

The log of a SUM is NOT the sum of logs. #1 mistake. ($\log_b(xy) = \log_b(x) + \log_b(y)$ IS true — Product Rule in 4.4.)

✗ **Forgetting the domain restriction**

You can only take log of a POSITIVE number. $\log(0)$ and $\log(-5)$ are undefined. Always check solutions!

✗ **Confusing base and argument**

In $\log_b(x) = y$: $b$ is the base, $x$ is the argument, $y$ is the exponent. Convert: $b^y = x$ (not $x^y = b$).

✗ **Thinking log means multiply**

$\log_2(8) \neq \log \cdot 2 \cdot 8$. The subscript 2 is the BASE. $\log_2(8) = 3$ because $2^3 = 8$.

✗ **Not checking for extraneous solutions**

When solving log equations, ALWAYS verify answers don't make any argument $\leq 0$. Reject any that do.

# KEY TAKEAWAYS

1. $\log_b(x) = y$ means $b^y = x$

   The logarithm IS the exponent. Converting between forms is the fundamental skill.

2. Common log $= \log_{10}$, Natural log $= \ln = \log_e$

   These are the only two bases on your calculator. All others need change of base.

3. Properties: $\log_b(1) = 0$, $\log_b(b) = 1$, $\log_b(b^x) = x$

   Plus the inverse property $b^{\text{log\_b(x)}} = x$ and the one-to-one property.

4. Domain of $\log_b(x)$ is $(0, \infty)$ — argument must be positive

   Vertical asymptote at $x = 0$. Graph passes through $(1, 0)$. Inverse of $b^x$.

5. Change of base: $\log_a(x) = \log(x)/\log(a) = \ln(x)/\ln(a)$

   Use this to evaluate any log with your calculator.

# PRACTICE PROBLEMS

Test your understanding — try these on your own!

**Write in exponential form: $\log_6(216) = 3$ and $\ln(1) = 0$**

*Apply the definition: $\log_b(x) = y \leftrightarrow b^y = x$.*

**Evaluate without a calculator: $\log_4(1/16)$ and $\log_{27}(3)$**

*Set up $b^y = x$ and find y. Remember fractional exponents!*

**Simplify: $\log(10^6)$, $e^{\ln 15}$, $\log_8(8^{2.5})$**

*Use the basic properties and inverse properties.*

**Find the domain of $f(x) = \log_3(5 - 2x)$**

*Set $5 - 2x > 0$ and solve the inequality.*

**Solve: $\log_3(2x + 1) = 4$**

*Convert to exponential form: $3^4 = 2x + 1$.*

**Use change of base to evaluate $\log_5(83)$**

*$\log_5(83) = \log(83)/\log(5)$. Round to 3 decimal places.*

# PRACTICE PROBLEM ANSWERS

✓

**1** *Exponential form*

$6^3 = 216 \qquad e^0 = 1$

**2** *Evaluate*

$\log_4(1/16) = -2 \qquad \log_{27}(3) = 1/3$

**3** *Simplify*

$6 \;;\; 15 \;;\; 2.5$

**4** *Domain*

Domain: $(-\infty, 5/2) \qquad$ VA: $x = 5/2$

**5** *Solve*

$3^4 = 2x + 1 \rightarrow 81 = 2x + 1 \rightarrow x = 40$

**6** *Change of base*

$\log(83)/\log(5) \approx 2.745$

**Check your work! Review any problems you missed before moving to Section 4.4.**