Section 4.2

Exponential Functions

College Algebra & Trigonometry

Lecture Notes with Worked Examples

LEARNING OBJECTIVES

By the end of this section, you will be able to:

1. Graph exponential functions and identify key features
2. Understand and work with the natural base e
3. Apply compound interest formulas (periodic & continuous)
4. Model real-world growth and decay with exponential functions

REVIEW: EXPONENT RULES

Before we begin — make sure you're comfortable with these:

Product Rule

$a^m \cdot a^n = a^{m+n}$

Quotient Rule

$a^m / a^n = a^{m-n}$

Power Rule

$(a^m)^n = a^{mn}$

Zero Exponent

$a^0 = 1 \quad (a \neq 0)$

Negative Exponent

$a^{-n} = 1/a^n$

Fractional Exponent

$a^{(1/n)} = \sqrt[n]{a}$

DEFINITION

Exponential Function

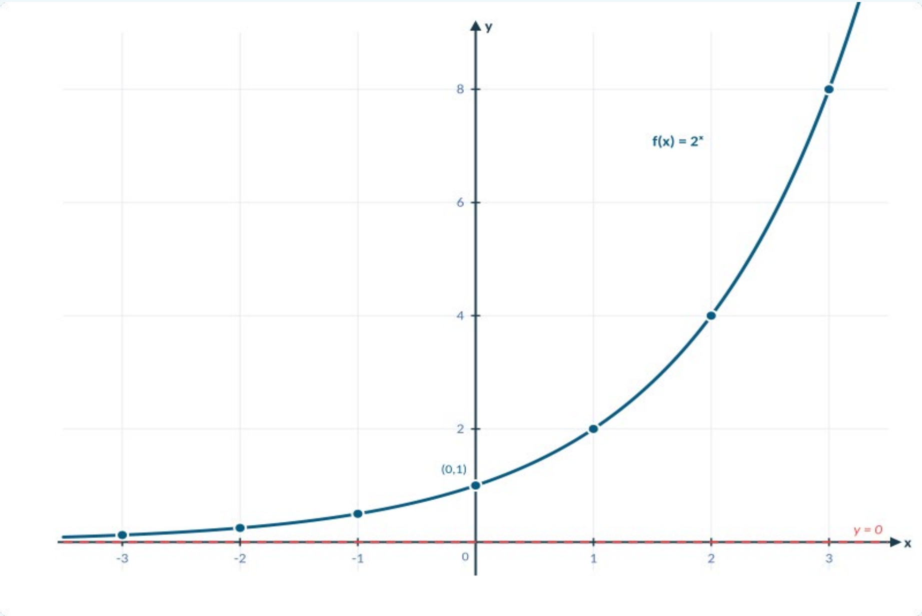
Let ***b*** be a constant with $b > 0$ and $b \neq 1$. Then for any real number x, the function $f(x) = b^x$ is an exponential function of base b.

Previously vs. Now

Power Function (variable base)	Exponential Function (variable exponent)
$f(x) = x^2$	$f(x) = 2^x$
Base varies, exponent is constant	Base is constant, exponent varies

EXAMPLE: GRAPHING $f(x) = 2^x$

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	⅛	¼	½	**1**	2	4	8



Key Observations

- y-intercept: (0, 1)
- Horizontal asymptote: $y = 0$
- Always positive (never touches x-axis)
- Increasing for all x ($b > 1$)
- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$

TRY IT: TABLE FOR AN EXPONENTIAL FUNCTION

The function f is defined by $f(x) = (\frac{1}{4})^x$. Find f(x) for each x-value in the table.

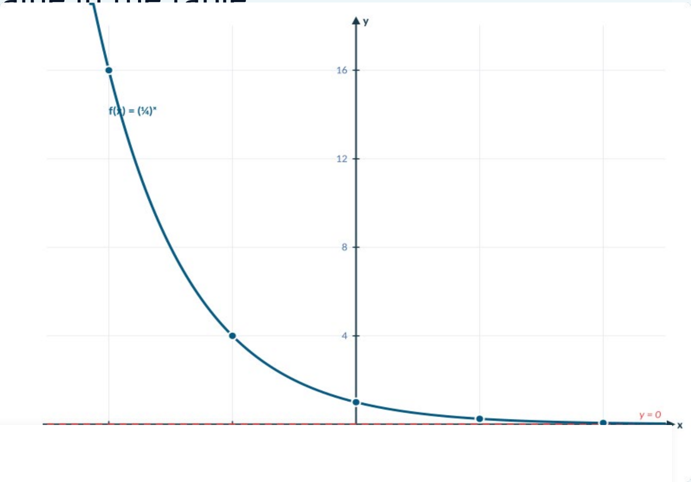
x	−2	−1	0	1	2
f(x)	?	?	?	?	?

Pause and try this before advancing!

TRY IT: TABLE FOR AN EXPONENTIAL FUNCTION

The function f is defined by $f(x) = (\frac{1}{4})^x$. Find f(x) for each x-value in the table.

x	−2	−1	0	1	2
f(x)	16	4	1	¼	1/16



Step-by-Step

$f(-2) = (\frac{1}{4})^{-2} = 4^2 = 16$ $f(-1) = (\frac{1}{4})^{-1} = 4^1 = 4$ $f(0) = (\frac{1}{4})^0 = 1$

$f(1) = (\frac{1}{4})^1 = 0.25$ $f(2) = (\frac{1}{4})^2 = 1/16 = 0.0625$

Pattern: As x increases, f(x) decreases — this is exponential decay (base $0 < b < 1$).

TRY IT: GRAPH $f(x) = 3^x$ AND ITS ASYMPTOTE

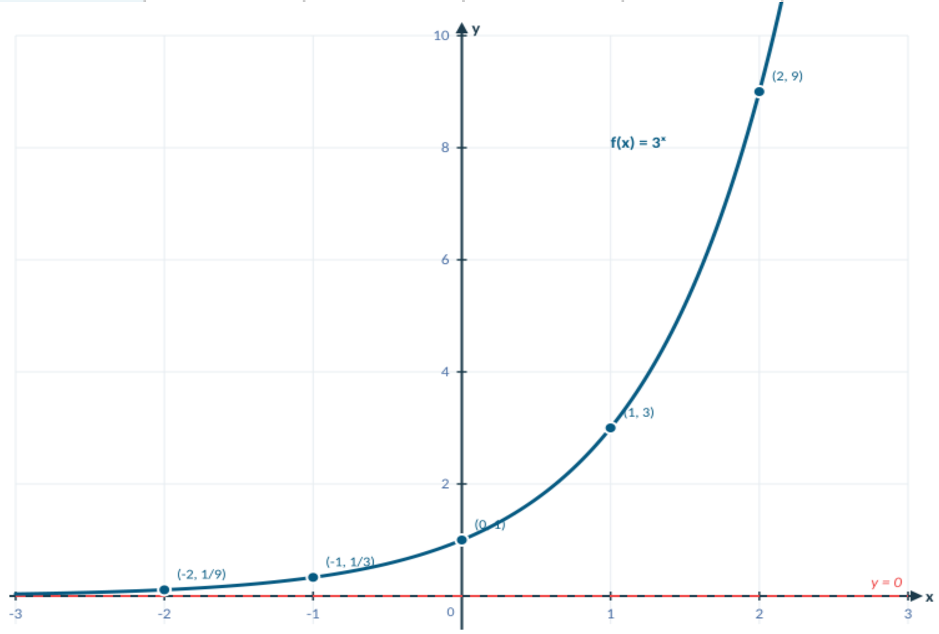
Plot five points on the graph of $f(x) = 3^x$ and draw the asymptote.

Build a table of values, plot the points, and identify the horizontal asymptote.

Pause and try this before advancing!

SOLUTION: GRAPH $f(x) = 3^x$

x	−2	−1	0	1	2
f(x)	1/9	1/3	1	3	9



Key Features

Five points:

(−2, 1/9), (−1, 1/3), (0, 1), (1, 3), (2, 9)

Horizontal asymptote:

y = 0 (the x-axis)

Domain: (−∞, ∞)

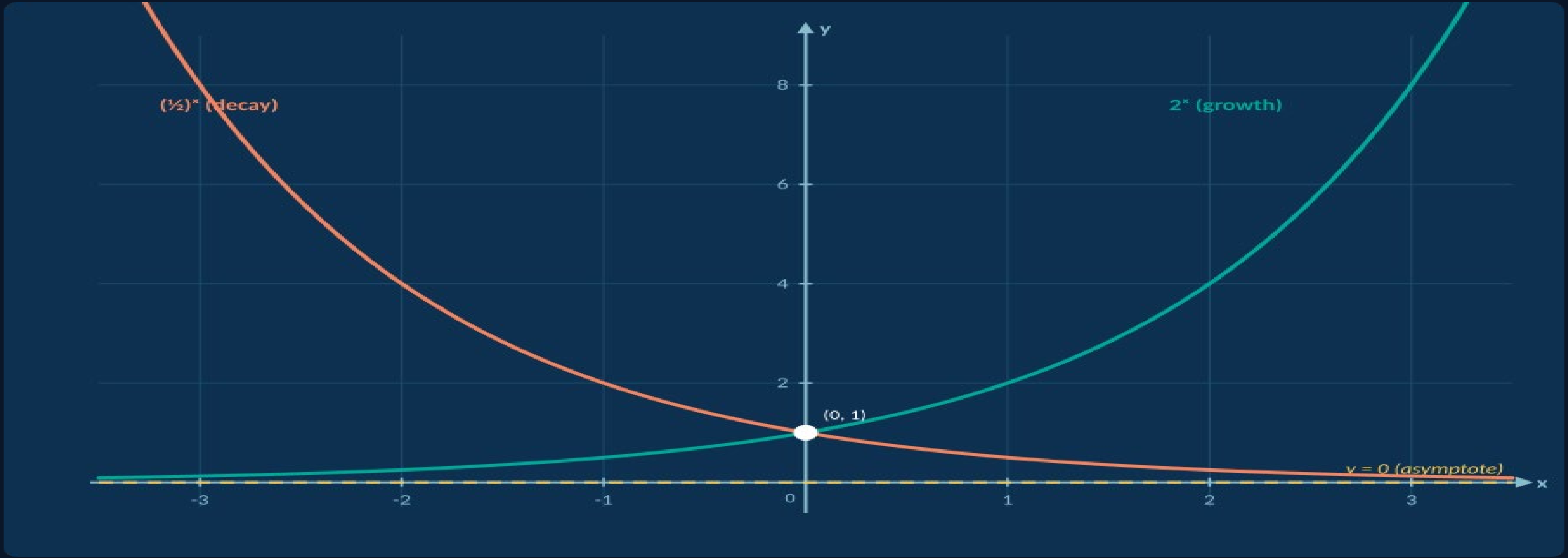
Range: (0, ∞)

y-intercept: (0, 1)

Increasing (b = 3 > 1)

Steeper than 2^x because the base is larger.

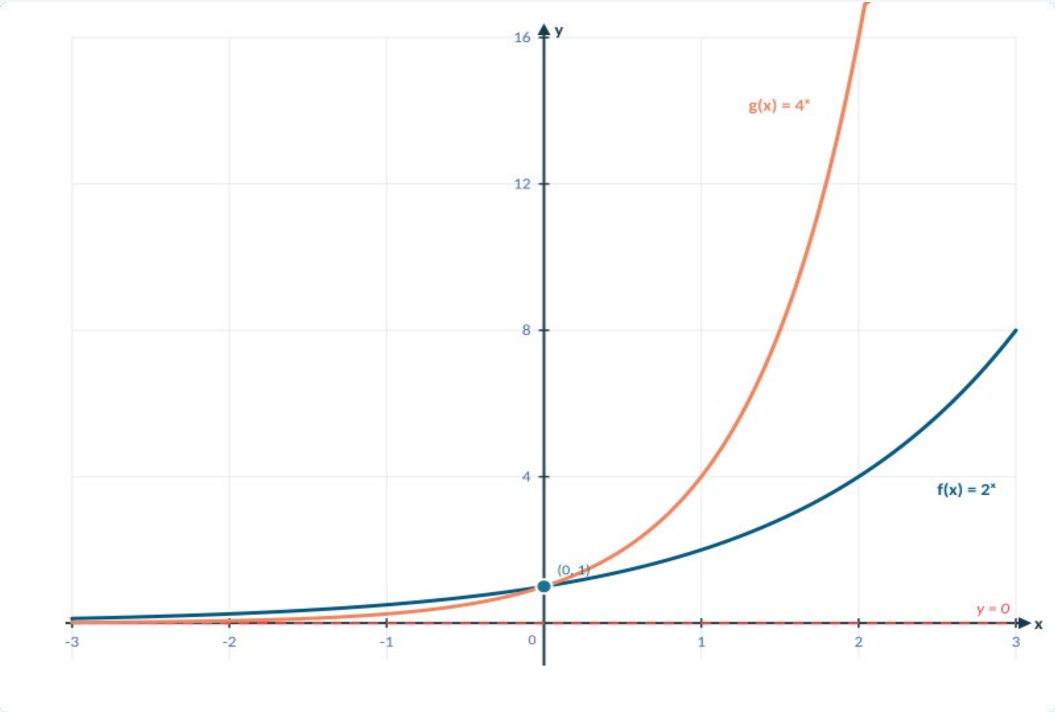
PROPERTIES OF $f(x) = b^x$



Both cases: Domain: $(-\infty, \infty)$ Range: $(0, \infty)$ y-intercept: $(0, 1)$ Asymptote: $y = 0$

COMPARING EXPONENTIAL FUNCTIONS

How does the graph of $f(x) = 2^x$ compare to $g(x) = 4^x$?



What to Notice

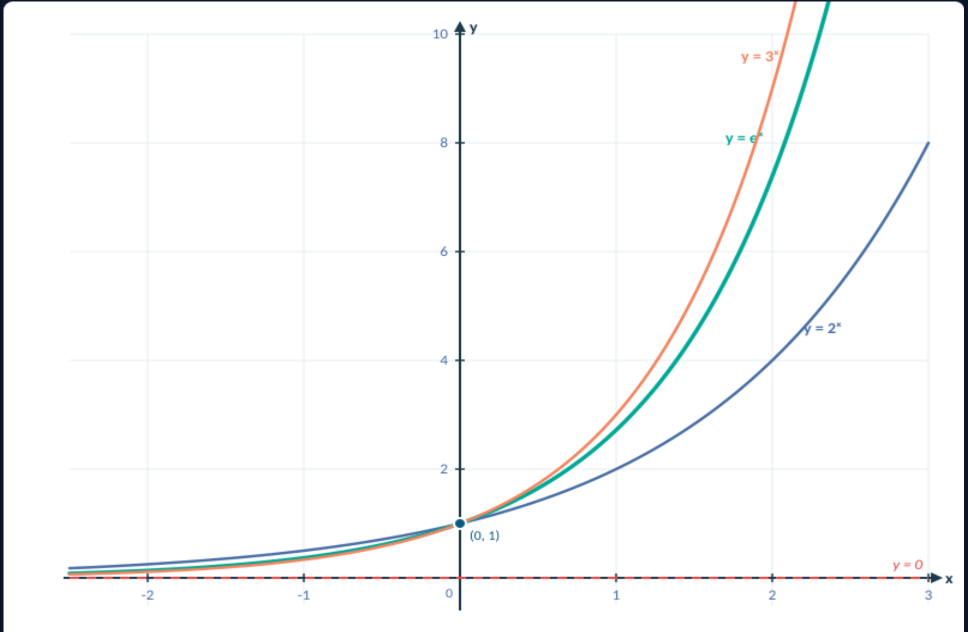
- Both pass through (0, 1)
- Larger base → steeper growth to the right
- Larger base → faster approach to 0 on the left
- Note: $4^x = (2^2)^x = 2^{2x}$, so 4^x is 2^x growing twice as fast

THE NATURAL BASE e

$e \approx 2.71828...$

irrational constant (like π)

As $n \to \infty$: $(1 + 1/n)^n \to e$



Watching $(1 + 1/n)^n$ converge:

n	1	10	100	1,000	100,000
$(1+1/n)^n$	2.000	2.594	2.705	2.717	2.718

TRY IT: EVALUATING EXPONENTIAL EXPRESSIONS

Use a calculator to evaluate each expression. Round to the nearest thousandth.

$$3.5^{1.6} = ?$$

$$\left(\tfrac{3}{4}\right)^{-0.95} = ?$$

Pause and try this before advancing!

SOLUTION: EVALUATING EXPONENTIAL EXPRESSIONS

$3.5^{1.6}$

Enter: 3.5 ^ 1.6 → **7.422**

$(3/4)^{-0.95}$

Enter: (3 ÷ 4) ^ (−0.95) →

Negative exponent flips: $(3/4)^{-0.95} = (4/3)^{0.95}$ (result > 1)

1.314

TRY IT: EVALUATING WITH BASE e

Use a calculator (2nd + LN = e^x). Round to the nearest thousandth.

$$e^{-0.15} = ?$$

$$175 \cdot e^{0.5} = ?$$

Pause and try this before advancing!

SOLUTION: EVALUATING WITH BASE e

$e^{-0.15}$

2nd → LN → (−0.15) → ENTER

$e^{-0.15} \approx 0.861$

$175 \cdot e^{0.5}$

175 × 2nd → LN → 0.5 → ENTER

$175 \cdot e^{0.5} \approx 288.526$

COMPOUND INTEREST FORMULAS

Suppose P dollars is invested (or borrowed) at annual interest rate r for t years.

SIMPLE INTEREST

$$I = Prt$$

P = principal
r = annual rate (decimal)
t = time in years
I = interest earned

PERIODIC COMPOUND

$$A = P(1 + r/n)^{nt}$$

A = future value
n = compounds per year
n=1 annually, n=4 quarterly
n=12 monthly, n=365 daily

CONTINUOUS

$$A = Pe^{rt}$$

As $n \to \infty$, periodic compounding → continuous

Uses base e

TRY IT: SIMPLE vs. COMPOUND INTEREST

Jenny: $70,000 at 3% compounded annually. Frank: $70,000 at 3% simple interest.
Compare the interest earned each year for 3 years.

Year	Jenny (Compound)	Frank (Simple)	Who earns more?
1st	?	?	?
2nd	?	?	?
3rd	?	?	?

SOLUTION: SIMPLE vs. COMPOUND INTEREST

Jenny: $70,000 at 3% compounded annually   Frank: $70,000 at 3% simple interest

Year	Jenny (Compound)	Frank (Simple)	Who earns more?
1st	**$2,100.00**	**$2,100.00**	**Same**
2nd	**$2,163.00**	$2,100.00	**Jenny**
3rd	**$2,227.89**	$2,100.00	**Jenny**

Why?

Year 1 — both earn the same because compound interest hasn't had time to accumulate.

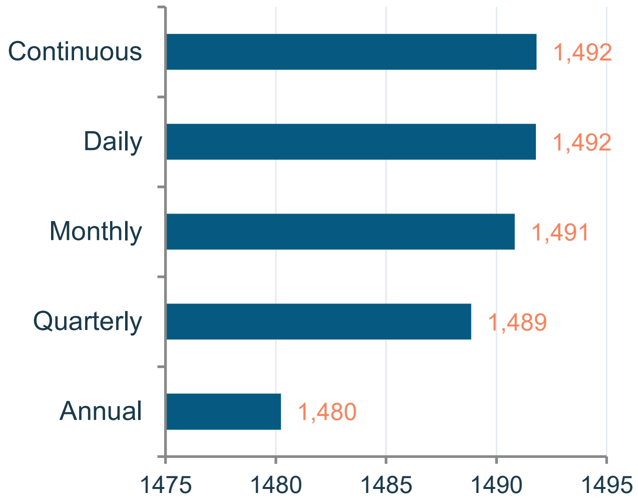
Year 2+ — Jenny earns interest on her interest (compounding effect). Frank's base never changes.

The gap grows every year — that's the power of compound interest!

EFFECT OF COMPOUNDING FREQUENCY

$1,000 invested at 4% for 10 years — how does compounding frequency affect the result?

Frequency	n	Formula	Amount	Interest Earned
Annual	1	$1000(1.04)^{10}$	**$1,480.24**	$480.24
Quarterly	4	$1000(1.01)^{40}$	**$1,488.86**	$488.86
Monthly	12	$1000(1.00333\ldots)^{120}$	**$1,490.83**	$490.83
Daily	365	$1000(1.0000110)^{3650}$	**$1,491.79**	$491.79
Continuous	∞	$1000 \cdot e^{0.4}$	**$1,491.82**	$491.82



Key Insight

More frequent compounding → higher return, but with diminishing gains.

Daily vs. continuous differs by only $0.03 — the jump from annual to quarterly matters most!

TRY IT: COMPOUND INTEREST — ANNUAL

$44,000 is borrowed for 13 years at 3.25% compounded annually. No payments. Find the amount owed. Round to the nearest dollar.

$$A = P(1 + r/n)^{nt}$$

P = ?	**r = ?**	**n = ?**	**t = ?**
Principal	Annual rate	Compounds/yr	Time in years

Identify P, r, n, and t, then substitute into the formula.

Pause and try this before advancing!

SOLUTION: $44,000 AT 3.25% FOR 13 YEARS

Identify Variables

$P = \$44{,}000$ $r = 0.0325$ $n = 1$ (annually) $t = 13$ years

Substitute & Solve

$$A = P(1 + r/n)^{nt}$$

$$A = 44{,}000(1 + 0.0325/1)^{1(13)} = 44{,}000(1.0325)^{13}$$

$$A = 44{,}000 \times 1.51554\ldots$$

$A \approx \$66{,}684$

TRY IT: COMPOUND INTEREST — SEMIANNUAL

$2,000 is invested at 3.1% compounded semiannually. Find the total after 3 years. Round to the nearest cent.

$$A = P(1 + r/n)^{nt}$$

Compounding Frequency Reference

Annually: $n = 1$ Semiannually: $n = 2$ Quarterly: $n = 4$ Monthly: $n = 12$ Daily: $n = 365$

Hint: What is n for semiannual? What is the total number of compounding periods (nt)?

Pause and try this before advancing!

SOLUTION: $2,000 AT 3.1% SEMIANNUALLY

Identify Variables

$P = \$2{,}000$ $r = 0.031$ $n = 2$ (semiannually) $t = 3$ years

Substitute & Solve

$A = 2{,}000(1 + 0.031/2)^{2(3)} = 2{,}000(1.0155)^6$

$A = 2{,}000 \times 1.09668\ldots$

$A \approx \$2{,}193.36$

TRY IT: CONTINUOUS COMPOUND INTEREST

$1,600 borrowed at 6.5% compounded continuously for 6 years. Find amount owed. Round to nearest cent.

$$A = Pe^{rt}$$

Which Formula Do I Use?

"compounded annually / quarterly / monthly / daily" → $A = P(1 + r/n)^{nt}$

"compounded continuously" → $A = Pe^{rt}$ (no n needed!)

Pause and try this before advancing!

SOLUTION: $1,600 CONTINUOUS AT 6.5%

Identify Variables

$P = \$1{,}600$ $r = 0.065$ $t = 6$ years (continuous → use $A = Pe^{rt}$)

Substitute & Solve

$A = Pe^{rt} = 1{,}600 \cdot e^{0.065(6)} = 1{,}600 \cdot e^{0.39}$

$A = 1{,}600 \times 1.47698\ldots$

$A \approx \$2{,}363.17$

TRY IT: RADIOACTIVE DECAY

Uranium-240 has a half-life of 14 hours. The amount remaining (in grams) after t hours:

$$A(t) = 3200 \cdot (½)^\wedge(t/14)$$

Find: (a) The initial amount (b) The amount remaining after 40 hours

Pause and try this before advancing!

SOLUTION: RADIOACTIVE DECAY

(a) Initial Amount (t = 0)

$A(0) = 3200 \cdot (\frac{1}{2})^{0/14}$

$= 3200 \cdot (\frac{1}{2})^0 = 3200 \cdot 1$

= 3,200 grams

The initial sample is 3,200 g.

(b) After 40 hours

$A(40) = 3200 \cdot (\frac{1}{2})^{40/14}$

$= 3200 \cdot (\frac{1}{2})^{2.857}$

$= 3200 \times 0.13811\ldots$

≈ 442 grams

After 40 hours, ≈ 442 g remain.

The coefficient in front (3200) is always the initial amount. Verify: plug in t = 0.

COMMON MISTAKES TO AVOID

Confusing x^2 with 2^x

x^2 = power function (variable base). 2^x = exponential (variable exponent). They grow at completely different rates.

Using $r = 5$ instead of $r = 0.05$

Always convert the percentage to a decimal first! Divide by 100. If your answer seems absurdly large, check this.

Wrong n value in $A = P(1+r/n)^{nt}$

n = 1 annually, 2 semiannually, 4 quarterly, 12 monthly, 365 daily. n is NOT the number of years (that's t).

Using $A = P(1+r/n)^{nt}$ for continuous

If the problem says "compounded continuously," use $A = Pe^{rt}$. There is no n — that's the point.

Forgetting parentheses on calculator

(3/4)^(−2) ≠ 3/4^(−2). Always parenthesize the base and the exponent. When in doubt, add more parentheses.

KEY TAKEAWAYS

1. **$f(x) = b^x$ is exponential when $b > 0$, $b \neq 1$**

   The variable is in the exponent — that's what makes it exponential.

2. **$b > 1 \rightarrow$ growth, $0 < b < 1 \rightarrow$ decay**

   All pass through (0, 1) with horizontal asymptote y = 0.

3. **$e \approx 2.71828$ is the natural base**

   Appears in continuous compounding, physics, biology, and calculus.

4. **$A = P(1 + r/n)^{nt}$ and $A = Pe^{rt}$**

   Periodic vs. continuous — more frequent → higher return (diminishing gains).

5. **Exponential models describe real-world phenomena**

   Radioactive decay, population growth, viral spread, and financial growth.

PRACTICE PROBLEMS

Test your understanding — try these on your own!

1. **Evaluate: 5^{-3} and $(⅔)^{-2}$**

   Apply exponent rules to simplify.

2. **Graph $f(x) = (⅓)^x$. Identify the asymptote, domain, range, and whether it shows growth or decay.**

   Build a table with $x = -2, -1, 0, 1, 2$.

3. **\$8,500 is invested at 4.2% compounded quarterly for 6 years. Find the account balance.**

   Use $A = P(1 + r/n)^{nt}$ with $n = 4$.

4. **An investment of \$12,000 earns 5.8% compounded continuously. Find the value after 10 years.**

   Use $A = Pe^{rt}$.

5. **A bacteria culture starts with 500 cells and doubles every 3 hours. Write the model B(t) and find the population after 10 hours.**

   Model: B(t) = 500 · 2^(t/3).